Solution to Assignment 3

Supplementary Problems

1. Evaluate the improper integral
$$\int_0^\infty e^{-x^2}\,dx\ .$$

Solution. Observe
$$\left(\int_0^\infty e^{-x^2}\,dx\right)^2 = \iint_S e^{-x^2-y^2},$$
where S is the positive quadrant $\{(x,y) : x, y \geq 0\}$. Switching to polar coordinates,
$$\iint_S e^{-x^2-y^2} = \int_0^{\pi/2}\int_0^\infty e^{-r^2} r\,drd\theta = \int_0^{\pi/2}\lim_{a\to\infty}\int_0^a e^{-r^2} r\,drd\theta = \frac{\pi}{4}\ .$$
It follows that
$$\int_0^\infty e^{-x^2}\,dx = \sqrt{\frac{\pi}{4}}\ .$$

2. Discuss the existence of the improper integral
$$\iint_D \frac{y}{(x^2+y^2)^{3/2}}\ ,$$
where D is the region enclosed by the polar graph $r = 1+\cos\theta$ and $y \geq 0$.

Solution. Let a be a small positive number. We consider
$$I(a) \equiv \iint_{D_a} \frac{y}{(x^2+y^2)^{3/2}}\ ,$$
where D_a is the region bounded between $r = 1+\cos\theta, y \geq 0$ and $r = a$. Using polar coordinates,
$$I(a) = \int_0^{\theta_0}\int_a^{1+\cos\theta} \frac{r\sin\theta}{r^3} r\,drd\theta\ ,$$
where θ_0 satisfies $1+\cos\theta_0 = a$. Hence
$$I(a) = \int_0^{\theta_0} \sin\theta(\log(1+\cos\theta) - \log a)\,d\theta\ .$$
On one hand, we have
$$\begin{aligned}\int_0^{\theta_0} \sin\theta\log(1+\cos\theta)\,d\theta &= -\int_1^{\cos\theta_0} \log(1+t)\,dt \\ &= -(1+\cos\theta_0)\log(1+\cos\theta_0) + \cos\theta_0 + 2\log 2 - 1 \\ &\to 2\log 2 - 2\ .\end{aligned}$$
as $a \to 0$ ($\theta_0 \to \pi$). On the other hand,
$$-\int_0^{\theta_0} \sin\theta\log a\,d\theta = \log a(\cos\theta_0 - 1) \to -\infty$$
as $a \to 0$. That is,
$$\lim_{a\to 0} I(a) = -\infty\ ,$$
the improper integral does not exist.